

Scalar Casimir densities induced by a cylindrical shell in de Sitter spacetime

A. A. Saharian^{1*}, V. F. Manukyan²

¹*Department of Physics, Yerevan State University,
1 Alex Manoogian Street, 0025 Yerevan, Armenia*

²*Department of Physics and Mathematics, Gyumri State Pedagogical Institute,
4 Paruyr Sevak Street, 3126 Gyumri, Armenia*

June 3, 2014

Abstract

We evaluate the positive-frequency Wightman function, the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor for a massive scalar field with general curvature coupling for a cylindrical shell in background of dS spacetime. The field is prepared in the Bunch-Davies vacuum state and on the shell the corresponding operator obeys Robin boundary condition. In the region inside the shell and for non-Neumann boundary conditions, the Bunch-Davies vacuum is a physically realizable state for all values of the mass and curvature coupling parameter. For both interior and exterior regions, the VEVs are decomposed into boundary-free dS and shell-induced parts. We show that the shell-induced part of the vacuum energy-momentum tensor has a nonzero off-diagonal component corresponding to the energy flux along the radial direction. Unlike to the case of a shell in Minkowski bulk, for dS background the axial stresses are not equal to the energy density. In dependence of the mass and of the coefficient in the boundary condition, the vacuum energy density and the energy flux can be either positive or negative. The influence of the background gravitational field on the boundary-induced effects is crucial at distances from the shell larger than the dS curvature scale. In particular, the decay of the VEVs with the distance is power-law (monotonic or oscillatory with dependence of the mass) for both massless and massive fields. For Neumann boundary condition the decay is faster than that for non-Neumann conditions.

PACS numbers: 04.62.+v, 03.70.+k, 11.10.Kk

1 Introduction

The study of the Casimir effect (for reviews see [1]-[5]) for geometries involving cylindrical boundaries have attracted considerable theoretical and experimental interest. In addition to traditional problems of quantum electrodynamics under the presence of material boundaries, the Casimir effect for cylindrical geometries can also be important to the flux tube models of confinement in quantum chromodynamics [6, 7] and for determining the structure of the vacuum state in interacting field theories [8]. A number of widely used nanostructures, such as single- and multi-walled carbon nanotubes, have cylindrical shapes. From the point of view of the experimental studies, the geometries

*E-mail: saharian@ysu.am

with cylindrical boundaries are among the most optimal candidates for the precision measurements of the Casimir force. Compared to the case of spherical boundaries, in these geometries the effective area of interaction is larger [9]-[11].

In view of this, the cylindrically symmetric boundary geometries are becoming increasingly important in the investigations of the Casimir effect. First the Casimir energy of an infinite perfectly conducting cylindrical shell has been evaluated in Ref. [12] on the base of a Green's function technique with an ultraviolet regulator. Later the corresponding result was rederived by the zeta function technique [13, 14] and by using the mode-by-mode summation technique [15] (for the Casimir energy and self-stresses in a more general problem of a dielectric-diamagnetic cylinder see [16] and references therein). The vacuum expectation value (VEV) of the energy-momentum tensor for the electromagnetic field in the interior and exterior regions of a conducting cylindrical shell are investigated in [17]. The geometry of two coaxial cylindrical shells is considered in [18] (see also [19]). The scalar Casimir densities and the vacuum energy for a single and two coaxial cylindrical shells with Robin boundary conditions are studied in [20, 21]. The zero-point energy of an arbitrary number of perfectly conducting coaxial cylindrical shells is calculated in [22] with the help of the mode summation technique. Less symmetric configuration of two eccentric cylinders is considered in [23] by using the mode summation and functional determinant methods. The Casimir self-energies for an elliptic cylinder are studied in [24]. The Casimir forces acting on two parallel plates inside a conducting cylindrical shell are investigated in [25]. The combined geometry of a wedge and coaxial cylindrical boundary is considered in [26]. The Casimir interaction energy in the configurations involving cylinders, plates and spheres has been discussed in [27] (see also [4, 5]).

In most studies of the Casimir effect with cylindrical boundaries the geometry of the background spacetime is Minkowskian. Combined effects of a cylindrical boundary and nontrivial topology induced by a cosmic string are discussed in [28]. For an idealized infinite straight cosmic string the spacetime is locally flat except on the top of the string where it has a delta shaped curvature tensor. In order to see the effects of the curvature on the Casimir densities induced by a cylindrical boundary, in the present paper we consider the background geometry described by de Sitter (dS) spacetime. The corresponding features for planar and spherically-symmetric boundaries are discussed in [29, 30]. The importance of dS background in gravitational physics is motivated by several reasons. First of all, dS spacetime is maximally symmetric and better understanding of physical effects on its backgrounds could serve as a handle to deal with more general geometries. The investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future. In most inflationary scenarios, the dS spacetime is employed to solve a number of problems in standard cosmology related to initial conditions in the early Universe. During an inflationary epoch the quantum fluctuations generate seeds for the formation of large scale structures in the Universe. More recently, cosmological observations have indicated that the expansion of the Universe at the present epoch is accelerating and the corresponding dynamics is well approximated by the model with a positive cosmological constant as a dominant source. For this source, the standard cosmology would lead to an asymptotic dS universe in the future.

We have organized the paper as follows. In the next section we evaluate the positive-frequency Wightman function for a scalar field with general curvature coupling inside and outside of a cylindrical shell on which the field obeys Robin boundary condition. We assume that the field is prepared in the Bunch-Davies vacuum state. The VEV of the field squared is investigated in section 3. The asymptotics are studied in detail at distances larger than the dS curvature scale. Section 4 is devoted to the investigation of the VEV of the energy-momentum tensor for both interior and exterior regions. We show that, in addition to the diagonal components, the vacuum energy-momentum tensor has an off-diagonal component which describes an energy flux along the radial direction. The main results are summarized and discussed in section 5.

2 Wightman function

We consider a quantum scalar field $\varphi(x)$ in background of a $(D+1)$ -dimensional dS spacetime, with Robin boundary condition (BC)

$$(A + Bn^l \nabla_l)\varphi(x) = 0, \quad (2.1)$$

imposed on a cylindrical shell having the radius a . Here, n^l is the normal to the shell, ∇_l is the covariant derivative operator, A and B are constants. Special cases of (2.1) correspond to Dirichlet ($B = 0$) and Neumann ($A = 0$) BCs. In accordance with the problem symmetry, we will write the dS line element in cylindrical coordinates (r, ϕ, \mathbf{z}) :

$$ds^2 = dt^2 - e^{2t/\alpha}[dr^2 + r^2 d\phi^2 + (d\mathbf{z})^2], \quad (2.2)$$

where $\mathbf{z} = (z^3, \dots, z^D)$. The Ricci scalar R and the corresponding cosmological constant Λ are expressed in terms of the parameter α as

$$R = D(D+1)\alpha^{-2}, \quad \Lambda = D(D-1)\alpha^{-2}/2. \quad (2.3)$$

In addition to the synchronous time coordinate t it is convenient to introduce the conformal time in accordance with

$$\tau = -\alpha e^{-t/\alpha}, \quad -\infty < \tau < 0. \quad (2.4)$$

The corresponding metric tensor is written in a conformally-flat form, $g_{ik} = \Omega^2 \eta_{ik}$, with the Minkowskian metric tensor η_{ik} and with the conformal factor $\Omega^2 = (\alpha/\tau)^2$.

For a free scalar field with a curvature coupling parameter ξ the field equation is in the form

$$(\nabla_l \nabla^l + m^2 + \xi R)\varphi(x) = 0. \quad (2.5)$$

For the special cases of minimally and conformally coupled fields one has the values of the curvature coupling $\xi = 0$ and $\xi = \xi_D = (D-1)/(4D)$, respectively. The imposition of boundary condition (2.1) on the field leads to modifications in the vacuum fluctuations spectrum and, as a result, to the change in the expectation values of the physical characteristics of the vacuum state $|0\rangle$. For a free field under consideration all the properties of the vacuum state are encoded in two-point functions. Here we will investigate the positive-frequency Wightman function, defined as the VEV $W(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$, assuming that the state $|0\rangle$ corresponds to the Bunch-Davies vacuum. Among the set of maximally symmetric quantum states in dS spacetime, the Bunch-Davies vacuum is the only one for which the ultraviolet behavior of the two-point functions is the same as in Minkowski spacetime.

The Wightman function can be presented in the form of the sum over a complete set of mode functions $\{\varphi_\sigma(x), \varphi_\sigma^*(x)\}$, obeying the field equation (2.5) and the boundary condition (2.1). The collective index σ will be specified below. The Wightman function is given by the expression

$$W(x, x') = \sum_{\sigma} \varphi_{\sigma}(x) \varphi_{\sigma}^*(x'). \quad (2.6)$$

Having this function we can evaluate the VEVs of the field squared and of the energy-momentum tensor. In addition, the Wightman function determines the transition rate of an Unruh-DeWitt particle detector in a given state of motion (see, for instance, [31]).

2.1 Interior region

First we consider the region inside the cylindrical shell, $r < a$. In the cylindrical spatial coordinates, for the corresponding mode functions, realizing the Bunch-Davies vacuum state, one has

$$\varphi_\sigma(x) = C_\sigma \eta^{D/2} H_\nu^{(2)}(\gamma\tau) J_n(\lambda r) e^{i(n\phi + \mathbf{k} \cdot \mathbf{z})}, \quad (2.7)$$

where $\eta = |\tau|$, $H_\nu^{(2)}(x)$ and $J_n(x)$ are the Hankel and Bessel functions, $\mathbf{k} = (k_3, \dots, k_D)$, $k = |\mathbf{k}|$, and we have defined

$$\begin{aligned} \nu &= [D^2/4 - D(D+1)\xi - \alpha^2 m^2]^{1/2}, \\ \gamma &= \sqrt{\lambda^2 + k^2}. \end{aligned} \quad (2.8)$$

With the choice (2.7), the collective index σ is specified to (n, λ, \mathbf{k}) , where $n = 0, \pm 1, \pm 2, \dots$, and $-\infty < k_l < +\infty$, $l = 3, \dots, D$. Note that the parameter ν can be either real or purely imaginary. For a conformally coupled massless field one has $\nu = 1/2$ and the Hankel function in (2.7) is expressed in terms of the exponential function. In this case the mode functions in dS spacetime are related to the corresponding functions for the shell in Minkowski spacetime by a conformal transformation.

The eigenvalues of the quantum number λ are determined from the boundary condition (2.1). Substituting the modes (2.7), we see that these eigenvalues are solutions to the equation

$$A J_n(\lambda a) + B \lambda J'_n(\lambda a) = 0, \quad (2.9)$$

where the prime means the derivative with respect to the argument of the function. For real A and B the roots of (2.9) are simple and real. We will denote the corresponding positive zeros by $\lambda a = \lambda_{n,l}$, $l = 1, 2, \dots$, assuming that they are arranged in ascending order: $\lambda_{n,l} < \lambda_{n,l+1}$. Now for the set of quantum numbers specifying the modes one has $\sigma = (n, l, \mathbf{k})$. Note that for Neumann BC ($A = 0$) the zero mode is present corresponding to $n = 0$, $\lambda = 0$.

The normalization coefficient C_σ in (2.7) is determined from the standard condition

$$\int d^D x \sqrt{|g|} g^{00} \varphi_\sigma(x) \overleftrightarrow{\partial}_\tau \varphi_{\sigma'}^*(x) = i \delta_{\sigma\sigma'}, \quad (2.10)$$

where the integration over the radial coordinate goes over the region inside the cylinder. In (2.10), $\delta_{\sigma\sigma'}$ stands for the Kronecker delta in the case of discrete components of σ (quantum numbers n and l) and for the Dirac delta function for continuous ones (\mathbf{k}). The normalization condition leads to the result

$$C_\sigma^2 = \frac{\lambda T_n(\lambda a) e^{i(\nu^* - \nu)\pi/2}}{4(2\pi)^{D-2} \alpha^{D-1} a}, \quad (2.11)$$

with the notation

$$T_n(z) = \frac{z}{(z^2 - n^2) J_n^2(z) + z^2 J_n'^2(z)}. \quad (2.12)$$

In the case of Neumann BC, the normalization coefficient for the zero mode is obtained from (2.11) putting $n = 0$ and taking the limit $\lambda \rightarrow 0$.

Substituting the eigenfunctions (2.7) into the mode sum formula (2.6) and by taking into account that $H_\nu^{(2)}(\gamma\tau) = (2i/\pi) e^{\nu\pi i/2} K_\nu(\gamma\eta e^{-\pi i/2})$, with $K_\nu(x)$ being the Macdonald function, for the Wightman function one finds the expression

$$\begin{aligned} W(x, x') &= \frac{4(\eta\eta')^{D/2}}{(2\pi)^D a \alpha^{D-1}} \sum_{n=-\infty}^{\infty} e^{in\Delta\phi} \int d\mathbf{k} e^{i\mathbf{k} \cdot \Delta\mathbf{z}} \sum_{l=1}^{\infty} \lambda T_n(\lambda a) \\ &\quad \times J_n(\lambda r) J_n(\lambda r') K_\nu(e^{-\pi i/2} \eta \gamma) K_\nu(e^{\pi i/2} \eta' \gamma) |_{\lambda=\lambda_{n,l}/a}, \end{aligned} \quad (2.13)$$

where $\Delta\phi = \phi - \phi'$, $\Delta\mathbf{z} = \mathbf{z} - \mathbf{z}'$, and γ is defined in (2.8). For the case of Neumann BC the contribution of the zero mode should be added to the right-hand side of (2.13). In order to separate explicitly the contribution induced by the cylindrical shell, we apply to the series over l the generalized Abel-Plana summation formula [32, 33]

$$2 \sum_{l=1}^{\infty} T_n(\lambda_{n,l}) f(\lambda_{n,l}) = \int_0^{\infty} dx f(x) + \frac{\pi}{2} \text{Res}_{z=0} f(z) \frac{\bar{Y}_n(z)}{J_n(z)} - \frac{1}{\pi} \int_0^{\infty} dx \frac{\bar{K}_n(x)}{\bar{I}_n(x)} \left[e^{-n\pi i} f\left(xe^{\pi i/2}\right) + e^{n\pi i} f\left(xe^{-\pi i/2}\right) \right], \quad (2.14)$$

where $f(z)$ is an analytic function on the right half-plane, $Y_n(z)$ is the Neumann function and, for a given function $F(z)$, we use the notation

$$\bar{F}(z) = AF(z) + (B/a)zF'(z). \quad (2.15)$$

As the function $f(z)$ in (2.14) we take

$$f(z) = zK_{\nu}(e^{-\pi i/2}\eta\sqrt{z^2/a^2 + k^2})K_{\nu}(e^{\pi i/2}\eta'\sqrt{z^2/a^2 + k^2})J_n(zr/a)J_n(zr'/a). \quad (2.16)$$

First let us consider the part in the Wightman function corresponding to the first term in the right-hand side of (2.14). We will denote it by $W_0(x, x')$:

$$W_0(x, x') = \frac{2(\eta\eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \sum_{n=-\infty}^{\infty} e^{in\Delta\phi} \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \int_0^{\infty} d\lambda \lambda \times J_n(\lambda r) J_n(\lambda r') K_{\nu}(e^{-\pi i/2}\eta\gamma) K_{\nu}(e^{\pi i/2}\eta'\gamma). \quad (2.17)$$

By using the relations

$$\sum_{n=-\infty}^{\infty} e^{in\Delta\phi} J_n(\lambda r) J_n(\lambda r') = J_0(\lambda\sqrt{r^2 + r'^2 - 2rr'\cos\Delta\phi}), \quad (2.18)$$

and

$$\int_0^{\infty} du u F(u) J_0(u\sqrt{z_1^2 + z_2^2}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 e^{ik_1 z_1 + ik_2 z_2} F(\sqrt{k_1^2 + k_2^2}), \quad (2.19)$$

one can see that

$$W_0(x, x') = \frac{2(\eta\eta')^{D/2}}{(2\pi)^{D+1} \alpha^{D-1}} \int d\mathbf{K} e^{i\mathbf{K}\cdot\Delta\mathbf{x}} K_{\nu}(e^{-\pi i/2}\eta|\mathbf{K}|) K_{\nu}(e^{\pi i/2}\eta'|\mathbf{K}|), \quad (2.20)$$

where $\mathbf{K} = (k_1, k_2, k_3, \dots, k_D)$, $\mathbf{x} = (z^1, z^2, \mathbf{z})$, $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$, $x = (\tau, \mathbf{x})$, $x' = (\tau, \mathbf{x}')$. After the evaluation of the integral (see, for instance, [34]), we get the expression

$$W_0(x, x') = \frac{\Gamma(D/2 + \nu)\Gamma(D/2 - \nu)}{(4\pi)^{(D+1)/2} \Gamma((D+1)/2) \alpha^{D-1}} F\left(\frac{D}{2} + \nu, \frac{D}{2} - \nu; \frac{D+1}{2}; w\right), \quad (2.21)$$

where

$$w = 1 + \frac{(\Delta\eta)^2 - |\Delta\mathbf{x}|^2}{4\eta\eta'}, \quad (2.22)$$

and $F(a, b; c; w)$ is the hypergeometric function. The function (2.21) is the Wightman function for the boundary-free dS spacetime.

Note that for $\text{Re } \nu \geq D/2$ the integral in (2.20) contains infrared divergences arising from long-wavelength modes. For these values of ν the Bunch-Davies vacuum state is not a physically realizable state in the boundary-free dS spacetime. The boundary conditions imposed on the field may exclude the modes leading to divergences. This is the case for the region inside the cylindrical shell with non-Neumann BCs. For these conditions, in (2.13) one has $\gamma \geq \lambda_{n,1}/a$ and the infrared divergences are absent regardless of ν . Consequently, the Bunch-Davies vacuum is a realizable state for all values of the parameter ν .

Now, after the application of (2.14) to the series over l in (2.13), we get the representation

$$W(x, x') = W_0(x, x') + W_b(x, x'), \quad (2.23)$$

where the contribution induced by the cylindrical shell is given by the expression

$$\begin{aligned} W_b(x, x') &= -\frac{4(\eta\eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \sum_{n=0}' \cos(n\Delta\phi) \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \\ &\times \int_0^\infty du u \frac{\bar{K}_n(ay)}{I_n(ay)} I_n(yr) I_n(yr') \mathcal{I}_\nu(u\eta, u\eta') \Big|_{y=\sqrt{u^2+k^2}}. \end{aligned} \quad (2.24)$$

with the function

$$\mathcal{I}_\nu(x, y) = I_{-\nu}(x) K_\nu(y) + K_\nu(x) I_\nu(y). \quad (2.25)$$

In (2.24), the prime on the sign of the summation means that the term $n = 0$ is taken with the coefficient $1/2$. The formula (2.24) is valid for $\text{Re } \nu < 1$ and in what follows we assume the values of ν in this range. Note that the contribution of the zero mode for Neumann BC is canceled by the second term in the right-hand side of (2.14). The representation (2.23) has two important advantages, compared to (2.13). First, the explicit knowledge of the roots $\lambda_{n,l}$ is not required. Second, the effects induced by the shell are explicitly separated and, for points away from the shell, the boundary-induced contribution is finite in the coincidence limit of the arguments. In this way, the renormalization of the VEVs of the field squared and the energy-momentum tensor is reduced to the one for the boundary-free dS spacetime. In addition, the integrand in (2.24) is an exponentially decreasing function at the upper limit of the integration instead of strongly oscillating function in (2.13).

2.2 Exterior region

In the exterior region, $r > a$, the radial part of the mode functions is a linear combination of the Bessel and Neumann functions. The relative coefficient in this combination is determined from the boundary condition (2.1) imposed on the shell. The mode functions realizing the Bunch-Davies vacuum state are written as

$$\varphi_\sigma(x) = C_\sigma \eta^{D/2} H_\nu^{(2)}(\tau\gamma) g_n(\lambda a, \lambda r) e^{i(n\phi + \mathbf{k}\cdot\mathbf{z})}, \quad (2.26)$$

where $0 \leq \lambda < \infty$ and

$$g_n(\lambda a, \lambda r) = \bar{Y}_n(\lambda a) J_n(\lambda r) - \bar{J}_n(\lambda a) Y_n(\lambda r), \quad (2.27)$$

with the notation defined by (2.15). Now, in (2.10) the integration over the radial coordinate goes over the region $a \leq r < \infty$ and in the right-hand side for the part corresponding to the quantum number λ one has $\delta(\lambda - \lambda')$. For the normalization coefficient we find

$$C_\sigma^2 = \frac{\lambda e^{i(\nu^* - \nu)\pi/2}}{8(2\pi)^{D-2} \alpha^{D-1}} [\bar{J}_n^2(\lambda a) + \bar{Y}_n^2(\lambda a)]^{-1}. \quad (2.28)$$

Substituting the functions (2.26) into the mode sum (2.6), and introducing the Macdonald function instead of the Hankel function, we get the following expression for the exterior Wightman function:

$$W(x, x') = \frac{2(\eta\eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \sum_{n=-\infty}^{\infty} e^{in\Delta\phi} \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \int_0^\infty d\lambda \lambda \times \frac{g_n(\lambda a, \lambda r) g_n(\lambda a, \lambda r')}{\bar{J}_n^2(\lambda a) + \bar{Y}_n^2(\lambda a)} K_\nu(e^{-\pi i/2} \eta \gamma) K_\nu(e^{\pi i/2} \eta' \gamma). \quad (2.29)$$

By using the identity

$$\frac{g_n(\lambda a, \lambda r) g_n(\lambda a, \lambda r')}{\bar{J}_n^2(\lambda a) + \bar{Y}_n^2(\lambda a)} = J_n(\lambda r) J_n(\lambda r') - \frac{1}{2} \sum_{j=1,2} \frac{\bar{J}_n(\lambda a)}{\bar{H}_n^{(j)}(\lambda a)} H_n^{(j)}(\lambda r) H_n^{(j)}(\lambda r'), \quad (2.30)$$

the Wightman function is presented in the decomposed form (2.23) with the shell-induced part

$$W_b(x, x') = -\frac{(\eta\eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \sum_{n=-\infty}^{\infty} e^{in\Delta\phi} \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \sum_{j=1,2} \int_0^\infty d\lambda \lambda \frac{\bar{J}_n(\lambda a)}{\bar{H}_n^{(j)}(\lambda a)} \times H_n^{(j)}(\lambda r) H_n^{(j)}(\lambda r') K_\nu(e^{-\pi i/2} \eta \gamma) K_\nu(e^{\pi i/2} \eta' \gamma). \quad (2.31)$$

Assuming that the functions $\bar{H}_n^{(1)}(z)$ and $\bar{H}_n^{(2)}(z)$ have no zeros for $0 < \arg z \leq \pi/2$ and $-\pi/2 \leq \arg z < 0$ respectively, we rotate the contour of integration over λ by the angle $\pi/2$ for the term with $j = 1$ and by the angle $-\pi/2$ for $j = 2$. The expression (2.31) takes the form

$$W_b(x, x') = -\frac{4(\eta\eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \sum_{n=0}^{\infty} \cos(n\Delta\phi) \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \times \int_0^\infty du u \frac{\bar{I}_n(ay)}{\bar{K}_n(ay)} K_n(ry) K_n(r'y) \mathcal{I}_\nu(u\eta, u\eta')|_{y=\sqrt{u^2+k^2}}. \quad (2.32)$$

Comparing with (2.24), we see that the boundary-induced part of the Wightman function in the exterior region is obtained from the corresponding expression in the interior region by the interchange $I_n \rightleftharpoons K_n$.

The expressions of the Wightman functions inside and outside a cylindrical shell in Minkowski spacetime are obtained from (2.24) and (2.32) in the limit $\alpha \rightarrow \infty$. In order to show that, we note that for large values of α one has $\nu \approx im\alpha$ and $\eta \approx \alpha - t$. By using the uniform asymptotic expansions of the functions $I_{\pm\nu}(|\nu|z)$ and $K_\nu(|\nu|z)$ (see [35]) for purely imaginary values of the order with a large modulus, it can be seen that the main contribution to the integrals comes from the range $u > m$ in which one has [30]

$$\mathcal{I}_\nu(u\eta, u\eta') \approx \frac{\cosh(\Delta t \sqrt{u^2 - m^2})}{\alpha \sqrt{u^2 - m^2}}, \quad (2.33)$$

with $\Delta t = t' - t$. Substituting into (2.24), for the Wightman function in the interior region we find

$$W_{M,b}(x, x') = -\frac{4}{(2\pi)^D} \sum_{n=0}^{\infty} \cos(n\Delta\phi) \int d\mathbf{k} e^{i\mathbf{k}\cdot\Delta\mathbf{z}} \int_{\sqrt{k^2+m^2}}^\infty dy y \times \frac{\cosh(\Delta t \sqrt{y^2 - k^2 - m^2})}{\sqrt{y^2 - k^2 - m^2}} \frac{\bar{K}_n(ay)}{\bar{I}_n(ay)} I_n(yr) I_n(yr'). \quad (2.34)$$

The expression in the exterior region is obtained by the interchange $I_n \rightleftharpoons K_n$. The corresponding VEVs for the both interior and exterior regions are investigated in [20].

3 VEV of the field squared

The VEVs of the field squared and of the energy-momentum tensor are among the most important characteristics of the vacuum state. The VEV of the field squared is obtained from the Wightman function by taking the coincidence limit of the arguments. Similar to the Wightman function, it is presented in the decomposed form

$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_0 + \langle \varphi^2 \rangle_b, \quad (3.1)$$

where $\langle \varphi^2 \rangle_0$ is the VEV in the boundary-free dS spacetime and the part $\langle \varphi^2 \rangle_b$ is induced by the cylindrical shell. The boundary-free part is widely investigated in the literature. Because of the maximal symmetry of the dS spacetime and of the Bunch-Davies vacuum state, the renormalized boundary-free VEV does not depend on the spacetime point. In what follows we will be concerned with the boundary-induced effects.

3.1 Interior region

In the region inside the shell, for the boundary-induced contribution from (2.24) one has

$$\langle \varphi^2 \rangle_b = -\frac{A_D}{\alpha^{D-1}} \sum_{n=0}^{\infty} \int_0^{\infty} du u^{D-1} \frac{\bar{K}_n(ua/\eta)}{\bar{I}_n(ua/\eta)} I_n^2(ur/\eta) h_\nu(u), \quad (3.2)$$

where

$$A_D = \frac{\pi^{-D/2-1}}{2^{D-3}\Gamma(D/2-1)}. \quad (3.3)$$

In (3.2), we have defined the function

$$h_\nu(u) = \int_0^1 ds s(1-s^2)^{D/2-2} f_\nu(us), \quad (3.4)$$

with

$$f_\nu(y) = [I_{-\nu}(y) + I_\nu(y)] K_\nu(y). \quad (3.5)$$

In deriving (3.2), we first integrated over the angular part of \mathbf{k} in (2.24) and then introduced polar coordinates in the (k, u) -plane. The integral in (3.4) is obtained from the integral over the polar angle. For points outside the shell, $r < a$, the boundary-induced part is finite and the renormalization is needed for the boundary-free part only.

The boundary-induced contribution to the VEV depends on η , a and r in the form of the ratios a/η and r/η . This property is a consequence of the maximal symmetry of dS spacetime. By taking into account that $\alpha a/\eta$ is the proper radius of the cylinder and $\alpha r/\eta$ is the proper distance from the cylinder axis, we see that a/η and r/η are the proper radius and the proper distance, measured in units of the dS curvature scale α . The function $f_\nu(y)$ is positive for $\nu \geq 0$. In this case, the part in the VEV of the field squared induced by the cylindrical shell is negative for Dirichlet BC and positive for Neumann BC. For purely imaginary values of ν the function $f_\nu(y)$ has no definite sign: it is positive for large values of the argument and oscillates in the region near $y = 0$.

For a conformally coupled massless field ($\xi = \xi_D$, $m = 0$), one has $\nu = 1/2$ and

$$f_\nu(y) = 1/y. \quad (3.6)$$

In this case, for the function $h_\nu(u)$ one has

$$h_\nu(u) = \frac{\sqrt{\pi}\Gamma(D/2-1)}{2\Gamma((D-1)/2)u}, \quad (3.7)$$

and from (3.2) we obtain

$$\langle \varphi^2 \rangle_b = \left(\frac{\eta}{\alpha} \right)^{D-1} \langle \varphi^2 \rangle_M, \quad (3.8)$$

where

$$\langle \varphi^2 \rangle_M = -\frac{2^{2-D} \pi^{-(D+1)/2}}{\Gamma((D-1)/2)} \sum_{n=0}^{\infty} \int_0^{\infty} du u^{D-2} \frac{\bar{K}_n(au)}{\bar{I}_n(au)} I_n^2(ru), \quad (3.9)$$

is the corresponding VEV inside a cylindrical shell in the Minkowski spacetime.

The boundary-induced part $\langle \varphi^2 \rangle_b$ diverges on the shell. The surface divergences in local physical characteristics of the vacuum state are well known in quantum field theory with boundaries. They are investigated for various types of bulk and boundary geometries. In the geometry under consideration, for points near the shell, the dominant contribution to (3.2) comes from large values of u and n . By taking into account that for large y one has $f_\nu(y) \approx 1/y$, we conclude that the leading term in the asymptotic expansion over the distance from the boundary coincides with that for a conformally coupled massless field. By taking into account the relation (3.8) and using the corresponding asymptotic for the shell in the Minkowski spacetime, we get

$$\langle \varphi^2 \rangle_b \approx -\frac{\Gamma((D-1)/2)(2\delta_{0B} - 1)}{(4\pi)^{(D+1)/2} [\alpha(a-r)/\eta]^{D-1}}. \quad (3.10)$$

In deriving (3.10) for $B \neq 0$ we have assumed that $a-r \ll |B|$. Note that $\alpha(a-r)/\eta$ is the proper distance from the shell. As it seen, near the shell the boundary-induced part in the VEV of the field squared is negative for Dirichlet BC ($B = 0$) and positive for non-Dirichlet BC.

On the axis of the shell the contribution of the term with $n = 0$ survives only and we get

$$\langle \varphi^2 \rangle_{b,r=0} = -\frac{A_D \alpha^{1-D}}{2(a/\eta)^D} \int_0^{\infty} dx x^{D-1} h_\nu(x\eta/a) \frac{\bar{K}_0(x)}{\bar{I}_0(x)}. \quad (3.11)$$

This expression is further simplified for large values of a/η corresponding to large values of the shell proper radius compared to the dS curvature radius. By using the asymptotic expressions for the modified Bessel functions with small values of the argument, to the leading order, for $u \ll 1$ one gets

$$h_\nu(u) \approx \frac{\Gamma(D/2 - 1)}{4} \sigma_\nu \text{Re} \left[\frac{(2/u)^{2\nu} \Gamma(\nu)}{\Gamma(D/2 - \nu)} \right], \quad (3.12)$$

where $\sigma_\nu = 1$ for positive ν and $\sigma_\nu = 2$ for purely imaginary ν . Substituting into (3.11), for positive ν we find

$$\langle \varphi^2 \rangle_{b,r=0} \approx -\frac{\pi^{-D/2-1} A^{(i)}(\nu)}{\alpha^{D-1} (2a/\eta)^{D-2\nu}}, \quad (3.13)$$

where

$$A^{(i)}(\nu) = \frac{\Gamma(\nu)}{\Gamma(D/2 - \nu)} \int_0^{\infty} dx x^{D-2\nu-1} \frac{\bar{K}_0(x)}{\bar{I}_0(x)}. \quad (3.14)$$

In this case $\langle \varphi^2 \rangle_{b,r=0}$ is a monotonic function of a/η .

For purely imaginary ν and for $a/\eta \gg 1$, the decay of the leading term is oscillatory:

$$\langle \varphi^2 \rangle_{b,r=0} \approx -\frac{2\pi^{-D/2-1} M^{(i)}}{\alpha^{D-1} (2a/\eta)^D} \cos[2|\nu| \ln(2a/\eta) + \phi^{(i)}], \quad (3.15)$$

where $M^{(i)}$ and $\phi^{(i)}$ are defined by the relation

$$A^{(i)}(\nu) = M^{(i)} e^{i\phi^{(i)}}, \quad (3.16)$$

with $M^{(i)} = |A^{(i)}(\nu)|$. For a given value of a , the expressions (3.13) and (3.15) describe the behavior of the VEV at late times of the expansion, $t \gg \alpha$. In the case of positive ν the shell-induced VEV on the axis decays as $e^{-(D-2\nu)t/\alpha}$, whereas for purely imaginary ν the decay is like $e^{-Dt/\alpha} \cos(\omega t + \phi^{(i)})$ with $\omega = 2|\nu|\alpha^{-1} \ln(2a/\eta)$.

3.2 Exterior region

In the region outside the cylindrical shell, taking the coincidence limit of the arguments in (2.32), for the boundary-induced part in the VEV of the field squared we get

$$\langle \varphi^2 \rangle_b = -\frac{A_D}{\alpha^{D-1}} \sum_{n=0}^{\infty} \int_0^{\infty} du u^{D-1} \frac{\bar{I}_n(ua/\eta)}{\bar{K}_n(ua/\eta)} K_n^2(ur/\eta) h_\nu(u), \quad (3.17)$$

with the function $h_\nu(u)$ defined in (3.4). For $\nu \geq 0$ the latter is positive and, similar to the case of the interior region, the shell-induced VEV is negative for Dirichlet BC and positive for Neumann BC. The expression in the right-hand side of (3.17) diverges on the shell. The leading term in the asymptotic expansion over the distance from the shell is given by expression (3.10) with $a - r$ replaced by $r - a$. In this region the effects of the curvature are small and the leading term coincides with that for the shell in Minkowski spacetime with the distance $r - a$ replaced by the proper distance $\alpha(r - a)/\eta$.

At large proper distances from the shell compared with the dS curvature radius we have $r/\eta \gg 1$ for a fixed value a/η . In this limit the dominant contribution to the integral in (3.17) comes from the region near the lower limit of the integration, $u \lesssim \eta/r$. For positive values of ν and for $A \neq 0$ the dominant contribution to (3.17) comes from the term $n = 0$. By taking into account that for small values of z one has $\bar{I}_0(z)/\bar{K}_0(z) \approx -1/\ln z$, to the leading order we find

$$\langle \varphi^2 \rangle_b \approx -\frac{\pi^{-(D+1)/2} \alpha^{1-D} A^{(e)}(\nu)}{4(2r/\eta)^{D-2\nu} \ln(r/a)}, \quad (3.18)$$

where

$$A^{(e)}(\nu) = \frac{\Gamma(\nu) \Gamma^2(D/2 - \nu)}{\Gamma(D/2 - \nu + 1/2)}. \quad (3.19)$$

Here, for $B \neq 0$ we have assumed that $r \gg |B|$. With this condition, the leading term does not depend on the values of the coefficients in the boundary condition and is negative. In the case of Neumann BC ($A = 0$) and for positive ν the leading contribution comes from the terms $n = 0$ and $n = 1$ with the asymptotic

$$\langle \varphi^2 \rangle_b \approx \frac{\alpha^{1-D} A_N^{(e)}(\nu) (a/\eta)^2}{2\pi^{(D+1)/2} (2r/\eta)^{D-2\nu+2}}, \quad (3.20)$$

where

$$A_N^{(e)}(\nu) = (3D/2 - 3\nu + 2) \frac{\Gamma(\nu) \Gamma^2(D/2 - \nu + 1)}{\Gamma(D/2 - \nu + 3/2)}. \quad (3.21)$$

For this case the decay of the boundary-induced part at large distances from the shell is faster and this part is positive. Combining with the asymptotic analysis for the region near the shell, we conclude that for Robin BC with $A, B \neq 0$ the shell-induced contribution in the VEV of the field squared is positive for points near the shell and negative at large distances. Hence, for some intermediate value of r it becomes zero. Note that at large distances the decay of the shell-induced VEV is power-law for both massless and massive fields. For a cylindrical shell in the Minkowski bulk and for a massive field the VEV of the field squared decays exponentially with the distance from the shell.

For purely imaginary values of the parameter ν and for $A \neq 0$ the leading asymptotic term is in the form

$$\langle \varphi^2 \rangle_b \approx \frac{\pi^{-(D+1)/2} \alpha^{1-D} M^{(e)}}{2(2r/\eta)^D \ln(a/r)} \cos[2|\nu| \ln(2r/\eta) + \phi^{(e)}], \quad (3.22)$$

where the constants $M^{(e)}$ and $\phi^{(e)}$ are defined by the relation

$$M^{(e)} e^{i\phi^{(e)}} = A^{(e)}(\nu). \quad (3.23)$$

For Neumann BC the asymptotic has the form

$$\langle \varphi^2 \rangle_b \approx \frac{\alpha^{1-D} (a/\eta)^2 M_N^{(e)}}{\pi^{(D+1)/2} (2r/\eta)^{D+2}} \cos[2|\nu| \ln(2r/\eta) + \phi_N^{(e)}], \quad (3.24)$$

with $M_N^{(e)}$ and $\phi_N^{(e)}$ defined as

$$M_N^{(e)} e^{i\phi_N^{(e)}} = A_N^{(e)}(\nu). \quad (3.25)$$

As we see, for imaginary ν the damping of the boundary-induced part with the distance from the shell is oscillatory.

4 Energy-momentum tensor

Now we turn to the investigation of the VEV for the energy-momentum tensor. In addition to describing the physical structure of a quantum field at a given point, it acts as the source of gravity in the quasiclassical Einstein equations and plays an important role in modeling self-consistent dynamics involving the gravitational field. Similar to the mean field squared, the VEV is decomposed as

$$\langle T_{ik} \rangle = \langle T_{ik} \rangle_0 + \langle T_{ik} \rangle_b, \quad (4.1)$$

where $\langle T_{ik} \rangle_0$ is the part corresponding to the boundary-free dS spacetime and $\langle T_{ik} \rangle_b$ is the boundary-induced part. From the maximal symmetry of dS spacetime and of the Bunch-Davies vacuum state it follows that the renormalized boundary-free part has the form $\langle T_i^k \rangle_0 = \text{const} \cdot \delta_i^k$. The boundary-induced contribution is obtained from the corresponding parts in the Wightman function and in the VEV of the field squared by using the formula

$$\langle T_{ik} \rangle_b = \lim_{x' \rightarrow x} \partial_i \partial'_k W_b(x, x') + [(\xi - 1/4) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik}] \langle \varphi^2 \rangle_b, \quad (4.2)$$

with $R_{ik} = Dg_{ik}/\alpha^2$ being the Ricci tensor for dS spacetime. In the right-hand side of (4.2) we have used the expression for the energy-momentum tensor of a scalar field which differs from the standard one (given, for example, in [31]) by a term which vanishes on the solutions of the field equation (2.5) (see [36]).

4.1 Interior region

First we consider the region inside the cylindrical shell. After lengthy but straightforward calculations, the VEVs for the diagonal components are presented in the form (no summation over i)

$$\begin{aligned} \langle T_i^i \rangle_b &= -\frac{A_D}{\alpha^{D+1}} \sum_{n=0}^{\infty} \int_0^{\infty} du u^{D+1} \frac{\bar{K}_n(ua/\eta)}{\bar{I}_n(ua/\eta)} \\ &\times \left\{ G_i[I_n(ur/\eta)] h_\nu(u) + I_n^2(ur/\eta) \int_0^1 ds s^3 (1-s^2)^{D/2-2} F_i(su) \right\}. \end{aligned} \quad (4.3)$$

In this formula we have defined the functions

$$\begin{aligned} G_0[f(z)] &= \left(\frac{1}{2} - 2\xi \right) \left[f'^2(z) + \left(1 + \frac{n^2}{z^2} \right) f^2(z) \right], \\ G_1[f(z)] &= -\frac{f'^2(z)}{2} - \frac{2\xi}{z} f(z) f'(z) + \frac{1}{2} \left(1 + \frac{n^2}{z^2} \right) f^2(z), \\ G_2[f(z)] &= G_0[f(z)] + \frac{2\xi}{z} f(z) f'(z) - \frac{n^2}{z^2} f^2(z) \\ G_l[f(z)] &= G_0[f(z)] - \frac{1}{D-2} f^2(z), \end{aligned} \quad (4.4)$$

with $l = 3, \dots, D$, and

$$\begin{aligned}
F_0(z) &= \left[\frac{1}{4} \partial_z^2 + \left(\frac{D+1}{4} - D\xi \right) \frac{1}{z} \partial_z + \frac{m^2 \alpha^2}{z^2} - 1 \right] f_\nu(z), \\
F_1(z) &= F_2(z) = \left(\xi - \frac{1}{4} \right) \partial_z^2 f_\nu(z) + \left[\xi(D+2) - \frac{D+1}{4} \right] \frac{1}{z} \partial_z f_\nu(z), \\
F_l(z) &= F_1(z) + \frac{1}{D-2} f_\nu(z).
\end{aligned} \tag{4.5}$$

Note that, unlike to the case of a shell in Minkowski bulk, here the stresses along the axial directions are not equal to the energy density.

In addition to the diagonal components, the shell-induced VEV of the energy-momentum tensor has also a nonzero off-diagonal component

$$\begin{aligned}
\langle T_0^1 \rangle_b &= -\frac{A_D}{2\alpha^{D+1}} \sum_{n=0}^{\infty} \int_0^\infty du u^D \frac{\bar{K}_n(ua/\eta)}{\bar{I}_n(ua/\eta)} I_n(ur/\eta) \\
&\quad \times I'_n(ur/\eta) [(1-4\xi)u\partial_u + D-4(D+1)\xi] h_\nu(u).
\end{aligned} \tag{4.6}$$

which corresponds to the energy flux along the radial direction.

The components of the energy-momentum tensor (4.3) and (4.6) are given in the coordinates $(\tau, r, \theta, \varphi)$. For the VEVs in the coordinates (t, r, θ, φ) with the synchronous time t , denoted here as $\langle T_{(s)i}^k \rangle_b$, one has the relations (no summation over i) $\langle T_{(s)i}^i \rangle_b = \langle T_i^i \rangle_b$ and $\langle T_{(s)0}^1 \rangle_b = (\eta/\alpha) \langle T_0^1 \rangle_b$. For a conformally coupled massless field, by using the relations (3.6) and (3.7), we can see that the off-diagonal component vanishes and the diagonal components of the vacuum energy-momentum tensor are related to the corresponding quantities inside a cylindrical shell in the Minkowski bulk, given in ([20]), by the conformal relation (no summation over i) $\langle T_i^i \rangle_b \approx (\eta/\alpha)^{D+1} \langle T_i^i \rangle_M$.

As an additional check of calculations, it can be seen that the shell-induced VEVs obey the covariant continuity equation, $\nabla_k \langle T_i^k \rangle_b = 0$, and the trace relation

$$\langle T_i^i \rangle_b = [D(\xi - \xi_D) \nabla_l \nabla^l + m^2] \langle \varphi^2 \rangle_b. \tag{4.7}$$

In particular, the shell-induced contribution is traceless for a conformally coupled massless field. The trace anomaly is contained in the boundary-free part of the VEV. The continuity equation is reduced to two relations between the components of the shell-induced part:

$$\begin{aligned}
\left(\partial_\tau - \frac{D+1}{\tau} \right) \langle T_0^0 \rangle_b + \frac{1}{\tau} \langle T_k^k \rangle_b + \left(\partial_r + \frac{1}{r} \right) \langle T_0^1 \rangle_b &= 0, \\
\left(\partial_\tau - \frac{D+1}{\tau} \right) \langle T_1^0 \rangle_b + \left(\partial_r + \frac{1}{r} \right) \langle T_1^1 \rangle_b - \frac{1}{r} \langle T_2^2 \rangle_b &= 0.
\end{aligned} \tag{4.8}$$

Note that $\langle T_1^0 \rangle_b = -\langle T_0^1 \rangle_b$.

The shell-induced part of the vacuum energy in the region $r \leq r_0 < a$, per unit coordinate lengths along the directions z^3, \dots, z^D , is given by

$$E_{r \leq r_0}^{(b)} = 2\pi (\alpha/\eta)^D \int_0^{r_0} dr r \langle T_0^0 \rangle_b. \tag{4.9}$$

For the corresponding time derivative from the first relation in (4.8) one gets

$$\partial_t E_{r \leq r_0}^{(b)} = \frac{2\pi}{\alpha} \left(\frac{\alpha}{\eta} \right)^D \int_0^{r_0} dr r \sum_{l=1}^D \langle T_l^l \rangle_b - 2\pi r_0 \left(\frac{\alpha}{\eta} \right)^{D-1} \langle T_0^1 \rangle_b|_{r=r_0}. \tag{4.10}$$

From here it is seen that $\langle T_0^1 \rangle_b$ is the energy flux per unit proper surface area. Note that $\langle T_l^l \rangle_b$ is the boundary-induced part of the vacuum pressure along the l -th direction. Equation (4.10) shows that the change of the energy is caused by two factors: by the work done by the surrounding (first term in the right-hand side of (4.10)) and by the energy flux through the boundary of the selected volume (second term).

Now let us discuss the asymptotics of the vacuum energy-momentum tensor. Near the cylindrical surface the dominant contribution to the boundary-induced VEVs comes from large values of n and u . By using the uniform asymptotic expansions for the modified Bessel functions (see, for instance, [37]), it can be seen that the leading terms in the diagonal components for a scalar field with non-conformal coupling ($\xi \neq \xi_D$) are related to the corresponding terms for a cylindrical boundary in Minkowski spacetime by (no summation over i) $\langle T_i^i \rangle_b \approx (\eta/\alpha)^{D+1} \langle T_i^i \rangle_M$. These leading terms are given by the expression (no summation over i)

$$\langle T_i^i \rangle_b \approx \frac{D\Gamma((D+1)/2)(\xi - \xi_D)}{2^D \pi^{(D+1)/2} [\alpha(a-r)/\eta]^{D+1}} (2\delta_{B0} - 1), \quad (4.11)$$

for the components with $i = 0, 2, \dots, D$. For the radial stress and the energy flux, to the leading order, one has

$$\langle T_1^1 \rangle_b \approx \frac{1-r/a}{D} \langle T_0^0 \rangle_b, \quad \langle T_0^1 \rangle_b \approx \frac{a-r}{\eta} \langle T_0^0 \rangle_b. \quad (4.12)$$

The leading terms have opposite signs for Dirichlet and non-Dirichlet BCs. In particular, for a minimally coupled field the energy density and the energy flux are negative for Dirichlet BC and positive for non-Dirichlet BC. Near the shell, the VEVs are dominated by the boundary-induced parts and the same is the case for the total energy density.

On the axis of the shell, $r = 0$, the only nonzero contribution to the diagonal components of the boundary-induced VEV comes from the terms in (4.3) with $n = 0, 1$. The energy flux vanishes on the axis as r/η . Simple expressions on the axis are obtained for large values of the shell proper radius compared with the dS curvature scale, $a/\eta \gg 1$. For positive values of ν , to the leading order we have (no summation over i)

$$\langle T_i^i \rangle_{b,r=0} \approx -\frac{\pi^{-D/2-1} A^{(i)}(\nu)}{\alpha^{D+1} (2a/\eta)^{D-2\nu}} A_i^i, \quad (4.13)$$

where $A^{(i)}(\nu)$ is defined by (3.14) and

$$\begin{aligned} A_0^0 &= D \left[\xi (2\nu - D - 1) + \frac{D - 2\nu}{4} \right], \\ A_l^l &= \frac{2\nu}{D} A_0^0, \quad l = 1, \dots, D. \end{aligned} \quad (4.14)$$

For a conformally coupled field one has $A_0^0 = (1 - 2\nu)/4$ and the leading term (4.13) vanishes in the massless case. For minimally and conformally coupled massive fields $A_0^0 > 0$. Now, by taking into account that $A^{(i)}(\nu) > 0$ for Dirichlet BC and $A^{(i)}(\nu) < 0$ for Neumann BC, we conclude that in these cases $\langle T_i^i \rangle_{b,r=0} < 0$ for Dirichlet BC and $\langle T_i^i \rangle_{b,r=0} > 0$ for Neumann BC.

For the energy flux in the limit $r \rightarrow 0$ and for $a/\eta \gg 1$, to the leading order one has

$$\langle T_0^1 \rangle_b \approx -\frac{4\pi^{-D/2-1} A_{10}^{(1)}(\nu) r/\eta}{\alpha^{D+1} (2a/\eta)^{D-2\nu+2}}, \quad (4.15)$$

with the function

$$A_{10}^{(1)}(\nu) = \frac{\Gamma(\nu) A_0^0}{D\Gamma(D/2 - \nu)} \int_0^\infty dx x^{D-2\nu+1} \left[\frac{\bar{K}_0(x)}{\bar{I}_0(x)} + \frac{\bar{K}_1(x)}{\bar{I}_1(x)} \right]. \quad (4.16)$$

As before, for a conformally coupled massless field the leading term vanishes. For minimally and conformally coupled massive fields the energy flux corresponding to (4.15) is negative for Dirichlet BC and positive for Neumann BC.

For purely imaginary ν and for $a/\eta \gg 1$, the behavior of the diagonal components on the axis is described by (no summation over i)

$$\langle T_i^i \rangle_{b,r=0} \approx -\frac{2\pi^{-D/2-1}M_i^{(i)}}{\alpha^{D+1}(2a/\eta)^D} \cos[2|\nu| \ln(2a/\eta) + \phi_i^{(i)}], \quad (4.17)$$

with $M_i^{(i)} = |A_i^{(i)}(\nu)A_i^i|$ and the phase $\phi_i^{(i)}$ defined by the relation

$$A_i^{(i)}(\nu)A_i^i = M_i^{(i)}e^{i\phi_i^{(i)}}. \quad (4.18)$$

For the energy flux near the axis, in the case of imaginary ν one has the leading term

$$\langle T_0^1 \rangle_b \approx -\frac{8\pi^{-D/2-1}M_{10}^{(i)}r/\eta}{\alpha^{D+1}(2a/\eta)^{D+2}} \cos[2|\nu| \ln(2a/\eta) + \phi_{10}^{(i)}], \quad (4.19)$$

where

$$A_{10}^{(i)}(\nu) = M_{10}^{(i)}e^{i\phi_{10}^{(i)}}. \quad (4.20)$$

For a given a , the expressions (4.13), (4.15), (4.17), (4.19) describe the asymptotic behavior of the shell-induced VEVs at late stages of the expansion, $t \gg \alpha$.

4.2 Exterior region

Now we turn to the investigation of the VEV for the energy-momentum tensor outside the cylindrical shell. The VEV is presented in the decomposed form (4.1) with the diagonal components of the boundary-induced part (no summation over i)

$$\begin{aligned} \langle T_i^i \rangle_b &= -\frac{A_D}{\alpha^{D+1}} \sum_{n=0}^{\infty} \int_0^{\infty} du u^{D+1} \frac{\bar{I}_n(ua/\eta)}{\bar{K}_n(ua/\eta)} \\ &\times \left\{ G_i[K_n(ur/\eta)]h_\nu(u) + K_n^2(ur/\eta) \int_0^1 ds (1-s^2)^{D/2-2} s^3 F_i(su) \right\}, \end{aligned} \quad (4.21)$$

where the functions $G_i[f(z)]$ and $F_i(z)$ are defined by (4.4) and (4.5). The off-diagonal component, corresponding to the energy flux along the radial direction, has the form

$$\begin{aligned} \langle T_0^1 \rangle_b &= -\frac{A_D}{2\alpha^{D+1}} \sum_{n=0}^{\infty} \int_0^{\infty} du u^D \frac{\bar{I}_n(ua/\eta)}{\bar{K}_n(ua/\eta)} K_n(ur/\eta) \\ &\times K_n'(ur/\eta) [(1-4\xi)u\partial_u + D-4(D+1)\xi] h_\nu(u). \end{aligned} \quad (4.22)$$

Similar to the case of the interior region, for the components of the vacuum energy-momentum tensor we have the relations (4.7) and (4.8).

On the boundary the VEVs diverge. The leading terms in the asymptotic expansion over the distance from the shell for the components $\langle T_i^i \rangle_b$ with $i = 0, 2, \dots, D$, are obtained from (4.11) by the replacement $(a-r) \rightarrow (r-a)$. Hence, these components have the same sign on both sides of the shell for points near the boundary. The leading terms for the radial stress, $\langle T_1^1 \rangle_b$, and the energy flux, $\langle T_0^1 \rangle_b$, are related to the energy density by (4.12). In particular, near the shell the radial stress and the energy flux for non-conformally coupled fields have opposite signs in the exterior and interior regions. For a minimally coupled field, the energy density near the shell is negative for

Dirichlet BC and positive for non-Dirichlet BC ($B \neq 0$). Near the boundary, for both interior and exterior regions, the corresponding energy flux is directed from the boundary for Dirichlet BC and to the boundary for non-Dirichlet BC.

Now we consider the asymptotic behavior of VEV for the energy-momentum tensor at large distances from the cylindrical shell, $r \gg a$. For the diagonal components the dominant contribution comes from the second term in the figure braces of (4.21). For the functions $F_i(z)$ in this term, for small values of the argument one has (no summation over i)

$$F_i(z) \approx \sigma_\nu \text{Re} \left[\frac{2^{2\nu-1} \Gamma(\nu)}{\Gamma(1-\nu)} A_i^i z^{-2\nu-2} \right], \quad (4.23)$$

with A_i^i defined by (4.14). By taking into account that at large distances from the shell the main contribution to the integrals in (4.21) and (4.22) comes from the region near the lower limit of the integral, for positive values of ν and for $A \neq 0$, to the leading order we get

$$\langle T_i^k \rangle_b \approx - \frac{\pi^{-(D+1)/2} \alpha^{-D-1} A_i^k A^{(e)}(\nu)}{2^{1+\delta_i^k} (2r/\eta)^{D-2\nu+1-\delta_i^k} \ln(r/a)}, \quad (4.24)$$

where $A^{(e)}(\nu)$ is given by (3.19) and

$$A_0^1 = (2\nu/D - 1) A_0^0. \quad (4.25)$$

The leading term (4.24) does not depend on the specific value of the ratio B/A . As is seen, at large distances, to the leading order, the shell-induced stresses are isotropic. For a conformally coupled massless field $A_0^0 = 0$ and the leading term given by (4.24) vanishes. In this case the diagonal components decay as $1/r^{D+2}$. For minimally and conformally coupled massive fields $A_0^0 > 0$ and the boundary-induced part in the energy density is negative at large distances. In accordance with (4.25), $\langle T_0^1 \rangle_b > 0$ and the energy flux is directed from the shell. For a cylindrical shell in Minkowski spacetime and for a massless field, at large distances the diagonal components decay as $r^{-D}/\ln(r/a)$ and the leading terms vanish for a conformally coupled field. For massive fields the decay of the Minkowskian VEVs with the distance from the shell is exponential.

For Neumann BC ($A = 0$) and for positive values of the parameter ν , at large distances, to the leading order, one has

$$\langle T_i^k \rangle_b \approx \frac{\pi^{-(D+1)/2} A_{Ni}^k A_N^{(e)}(\nu) (a/\eta)^2}{2\alpha^{D+1} (2r/\eta)^{D-2\nu+3-\delta_i^k}}, \quad (4.26)$$

where $A_{Ni}^i = A_i^i$ (no summation over i) and

$$A_{N0}^1 = -2(D - 2\nu + 2) A_0^0/D. \quad (4.27)$$

As before, for a conformally coupled massless field the leading term vanishes. For minimally and conformally coupled massive fields the boundary-induced part in the energy density is positive. In these cases $\langle T_0^1 \rangle_b < 0$ and the energy flux is directed toward the shell.

Combining with the results of the asymptotic analysis for the region near the shell, we conclude that for positive values of ν and for minimally and conformally coupled massive fields the shell-induced contribution in the VEV of the energy density is negative/positive near the shell and at large distances for Dirichlet/Neumann BC. For Robin BC with $A, B \neq 0$ this contribution is positive near the shell and negative at large distances. The energy flux is positive/negative for Dirichlet/Neumann BC near the shell and at large distances. For Robin BC with $A, B \neq 0$ the energy flux is negative near the shell and positive at large distances. At some intermediate value the energy flux vanishes.

Asymptotic behavior of the vacuum energy-momentum tensor at large distances from the shell is qualitatively different for imaginary values of ν . In this case and for non-Neumann BC ($A \neq 0$) the leading term has the form

$$\langle T_i^k \rangle_b \approx -\frac{\pi^{-(D+1)/2} \alpha^{-D-1} M_i^{(e)k}}{2^{\delta_i^k} (2r/\eta)^{D+1-\delta_i^k} \ln(r/a)} \cos[2|\nu| \ln(2r/\eta) + \phi_i^{(e)k}], \quad (4.28)$$

where the coefficient $M_i^{(e)k}$ and the phase $\phi_i^{(e)k}$ are defined by the relation

$$M_i^{(e)k} e^{i\phi_i^{(e)k}} = A_i^k A^{(e)}(\nu). \quad (4.29)$$

The decay of the VEVs is oscillatory. For Neumann BC and in the case of imaginary ν for the leading term in the asymptotic expansion one has

$$\langle T_i^k \rangle_b \approx \frac{2\alpha^{-D-1} M_{Ni}^{(e)k} (a/\eta)^2}{\pi^{(D+1)/2} (2r/\eta)^{D+3-\delta_i^k}} \cos[2|\nu| \ln(2r/\eta) + \phi_{Ni}^{(e)k}], \quad (4.30)$$

where

$$M_{Ni}^{(e)k} e^{i\phi_{Ni}^{(e)k}} = A_{Ni}^k A_N^{(e)}(\nu). \quad (4.31)$$

As is seen, for Neumann BC the suppression of the VEVs at large distances is faster. For a fixed value of r , the asymptotic expressions given above describe the behavior of the shell-induced contributions to the VEV of the energy-momentum tensor at late stages of the expansion corresponding to $t \gg \alpha$.

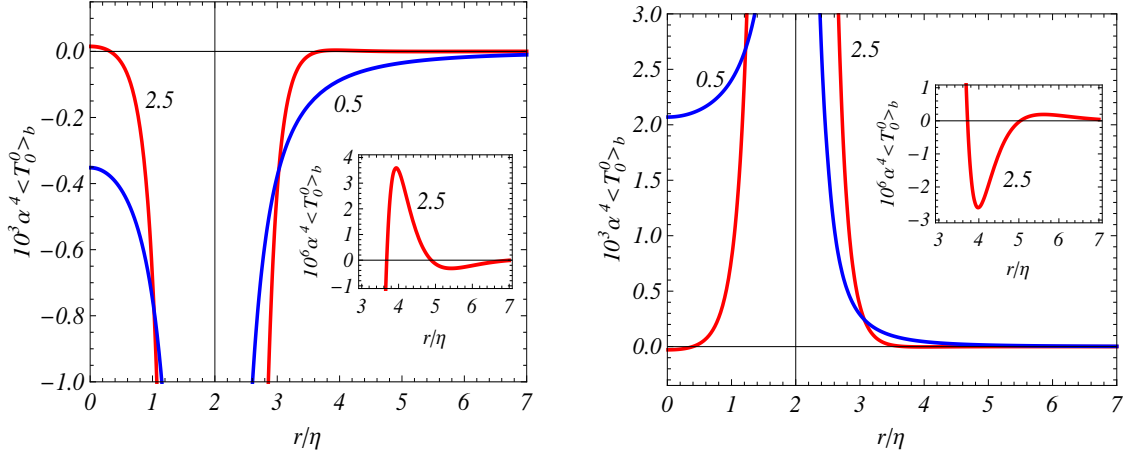


Figure 1: Shell-induced contribution in the VEV of the energy density as a function of the proper distance from the axis in the case of a conformally coupled field in $D = 3$ spatial dimensions. The numbers near the curves are the values of $m\alpha$ and we have taken $a/\eta = 2$. The left/right panels correspond to Dirichlet/Neumann BCs.

In figure 1, for a $D = 3$ conformally coupled field, we display the boundary-induced part in the energy density as a function of the proper distance from the cylindrical shell axis (measured in units of the dS curvature scale α). The graphs are plotted for the radius of the shell corresponding to $a/\eta = 2$ and the numbers near the curves are the values of the parameter $m\alpha$. The left/right panel corresponds to Dirichlet/Neumann BC. For $m\alpha = 2.5$ the parameter ν is purely imaginary and the oscillatory behavior at large distances from the shell is seen. Similar to the minimal coupling, for a conformally coupled field the energy density near the shell is negative for Dirichlet BC and positive for Neumann BC.

The same graphs for the energy flux are plotted in figure 2. Inside the shell, the energy flux is negative for Dirichlet BC and positive for Neumann BC. This means that the flux is directed from the shell for the first case and toward the shell in the second case. On the shell axis the energy flux vanishes as r/η .

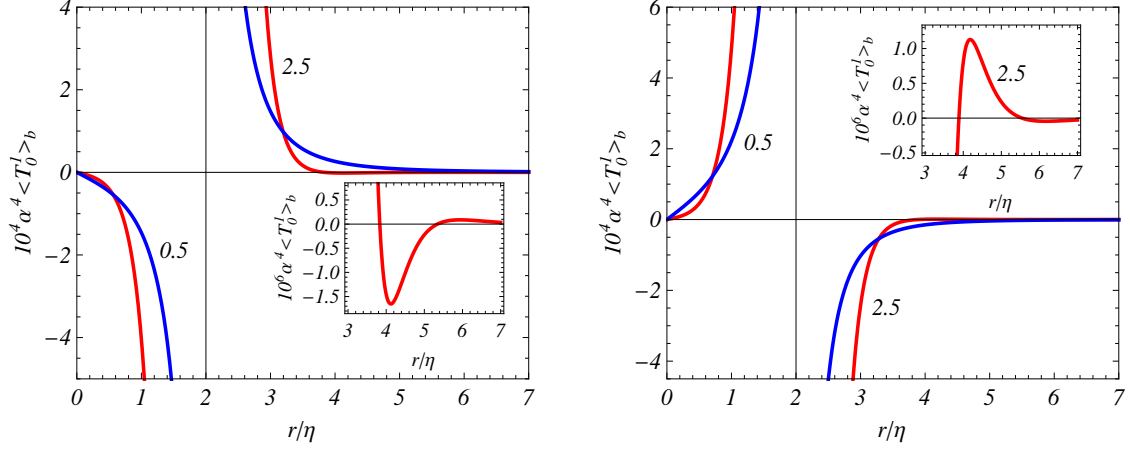


Figure 2: The same as in figure 1 for the energy flux.

Figure 3 shows the dependence of the boundary-induced parts in the VEVs of the energy density (left panel) and the energy flux (right panel) on the mass for a conformally coupled field in $D = 3$. The graphs are plotted for $a/\eta = 2$ and for fixed values of $r/\eta = 0.5$ and $r/\eta = 3$ (numbers near the curves). The full/dashed curves correspond to Dirichlet/Neumann boundary conditions. As it is seen from the presented examples, the VEVs for massive fields can be essentially larger than those in the massless case.

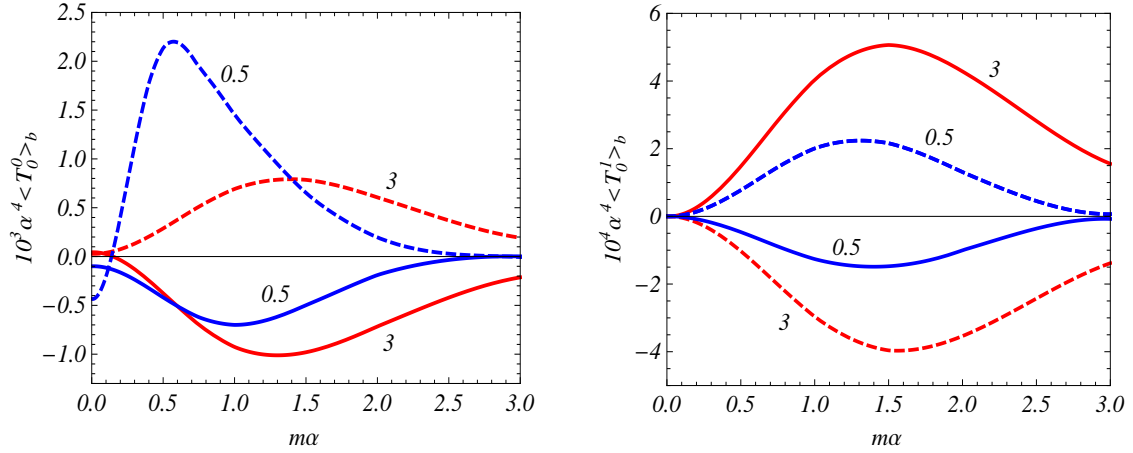


Figure 3: Shell-induced contributions in the VEVs of the energy density (left panel) and the energy flux (right panel) versus the mass for a $D = 3$ conformally coupled field. The graphs are plotted for $a/\eta = 2$ and for fixed values of $r/\eta = 0.5$ and $r/\eta = 3$ (numbers near the curves). The full/dashed curves correspond to Dirichlet/Neumann BCs.

5 Conclusion

In the present paper, for a free scalar field with general curvature coupling, we have investigated the change in the properties of the vacuum state induced by a cylindrical shell in background of dS spacetime. The Robin BC is imposed on the shell, which includes Dirichlet and Neumann BCs as special cases. We have assumed that the field is prepared in the Bunch-Davies vacuum state. In the region inside the shell and for non-Neumann BCs the zero mode is absent and in this region the Bunch-Davies vacuum is a physically realizable state for all values of the mass. All the properties of the vacuum are encoded in two-point functions. As such a function we have taken the positive-frequency Wightman function. Our method for the evaluation of this function employs the mode summation and, for the interior region, is based on a variant of the generalized Abel-Plana formula. This enabled us to extract explicitly the boundary-free dS part and to present the contribution induced by the shell in terms of strongly convergent integrals. The latter is given by the expression (2.24) in the interior region and by (2.32) for the exterior region. In the limit $\alpha \rightarrow \infty$, by using the uniform asymptotic expansions for the modified Bessel functions, the Wightman function is obtained for a shell in Minkowski spacetime.

Having the Wightman function, we have evaluated the VEVs of the field squared and of the energy-momentum tensor. These VEVs are decomposed into boundary-free dS and shell-induced parts. The boundary-free parts are widely discussed in the literature and we have concerned with the boundary-induced effects. For points outside the shell, the shell-induced parts of the VEVs are finite and the renormalization is reduced to that for the boundary-free geometry. The shell-induced contributions depend on the variables η , a , r through the ratios a/η and r/η which are the proper radius and the proper distance from the shell axis, measured in units of the dS curvature scale. This property is a consequence of the maximal symmetry of dS spacetime and of the Bunch-Davies vacuum state.

The shell-induced parts in the VEV of the field squared is given by expressions (3.2) and (3.17) for the interior and exterior regions respectively. For $\nu \geq 0$ the VEV is negative for Dirichlet BC and positive for Neumann BC in both regions. The boundary-induced part diverges on the shell with the leading term given by (3.10) for the interior region (in the exterior region $a - r$ should be replaced by $r - a$). For points near the shell the effects of the curvature are subdominant and the leading term coincides with that for the shell in Minkowski bulk. On the axis of the shell the term $n = 0$ contributes only and one has formula (3.11). Simple expressions are obtained for large values of the shell proper radius compared to the dS curvature scale, $a/\eta \gg 1$. For positive ν , on the axis, the shell-induced VEV of the field squared behaves like $(a/\eta)^{2\nu-D}$, whereas for purely imaginary ν the corresponding behavior, as a function of a/η , is damping oscillatory (see (3.15)). In the exterior region, at proper distances from the shell larger than the curvature radius of the background spacetime, the influence of the gravitational field on the boundary-induced VEVs is crucial. For positive values of ν and for non-Neumann BC the shell-induced part in the VEV of the field squared decays as $(r/\eta)^{2\nu-D}/\ln(r/a)$. For Neumann BC the decay is stronger, like $(r/\eta)^{2\nu-D-2}$. In the exterior region and for Robin BC with $A, B \neq 0$, the shell contribution in the mean field squared is positive for points near the shell and negative at large distances. For purely imaginary values of ν the behavior of the boundary-induced VEV at large distances is damping oscillatory and the leading term is given by (3.18) for non-Neumann BC and by (3.24) for Neumann BC. As before, the damping for Neumann BC is faster.

The diagonal components of the shell-induced contribution in the VEV of the energy-momentum tensor are given by expression (4.3) inside the shell and by (4.21) outside the shell. In addition to the diagonal components, the vacuum energy-momentum tensor has nonzero off-diagonal component given by the expressions (4.6) and (4.22) for the interior and exterior regions respectively. This component describes the energy flux along the radial direction and, in dependence of the parameters, it can be either positive or negative. Note that unlike to the case of a shell in Minkowski bulk, for

dS background the axial stresses are not equal to the energy density. Near the shell the leading term in the expansion of the energy density and of the stresses parallel to the shell is given by (4.11). For non-conformally coupled fields these VEVs diverge as the inverse $(D + 1)$ -th power of the proper distance from the boundary and near the shell they have the same sign in the interior and exterior regions. The leading terms for the normal stress and energy flux are given by (4.12) and the corresponding divergences are weaker. Near the shell, these components have opposite signs for the interior and exterior regions. For a minimally coupled field the energy flux near the shell is directed from the shell for Dirichlet BC and toward the shell for Neumann BC.

On the shell axis and for large values of the shell radius compared to the dS curvature scale, the diagonal components decay as $(a/\eta)^{2\nu-D}$ for positive values of ν and exhibit a damping oscillatory behavior, given by (4.17), for imaginary ν . The energy flux vanishes on the axis as r/η for $r \rightarrow 0$ and the corresponding asymptotic expressions are given by (4.15) and (4.19) for positive and imaginary ν , respectively. At distances from the shell larger than the dS curvature scale, to the leading order, the vacuum stresses are isotropic. For non-Neumann BCs the leading terms in the asymptotic expansion of the VEV of the energy-momentum tensor are given by (4.24) and (4.28) for positive and imaginary values of ν . In the first case the diagonal components decay as $(r/\eta)^{2\nu-D}/\ln(r/a)$ and the asymptotic for the energy flux contains an additional suppression factor η/r . For imaginary ν the decay of the VEVs is oscillatory. This type of behavior is a gravitationally induced effect and is absent in Minkowski bulk. For non-Neumann BCs the leading terms in the asymptotic expansions at large distances do not depend on the specific values of the coefficients in Robin BC. For minimally and conformally coupled massive fields and for positive ν , the shell-induced contribution in the energy density is negative at large distances and the energy flux is directed from the shell. For Neumann BC, the leading terms in the asymptotic expansion at large distances are given by expressions (4.26) and (4.30) for positive and imaginary ν respectively. In this case the decay of the VEVs at large distances is faster (by an additional factor $(\eta/r)^2$) than that for non-Neumann BCs. For minimally and conformally coupled massive fields with Neumann BC and for positive values of ν , the shell-induced contribution in the energy density is positive and the energy flux is directed toward the shell. For Robin BC with $A, B \neq 0$, the shell-induced energy density in the exterior region is positive near the shell and negative at large distances, whereas the energy flux is negative near the shell and positive at large distances. At some intermediate value of the radial coordinate these quantities vanish.

Acknowledgments

AAS was supported by State Committee Science MES RA, within the frame of the research project No. SCS 13-1C040.

References

- [1] V.M. Mostepanenko, N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997).
- [2] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
- [3] K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002).
- [4] M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

- [5] *Casimir Physics*, Lecture Notes in Physics Vol. 834, edited by D. Dalvit, P. Milonni, D. Roberts, F. da Rosa (Springer, Berlin, 2011).
- [6] P.M. Fishbane, S.G. Gasiorowich, P. Kauss, Phys. Rev. D **36**, 251 (1987); P.M. Fishbane, S.G. Gasiorowich, P. Kauss, Phys. Rev. D **37**, 2623 (1988).
- [7] B.M. Barbashov, V.V. Nesterenko, *Introduction to the Relativistic String Theory* (World Scientific, Singapore, 1990).
- [8] J. Ambjørn, S. Wolfram, Ann. Phys. **147**, 1 (1983).
- [9] M. Brown-Hayes, D.A.R. Dalvit, F.D. Mazzitelli, W.J. Kim, R. Onofrio, Phys. Rev. A **72**, 052102 (2005).
- [10] R.S. Decca, E. Fischbach, G.L. Klimchitskaya, D.E. Krause, D. López, V.M. Mostepanenko, Phys. Rev. A **82**, 052515 (2010).
- [11] E. Noruzifar, T. Emig, U. Mohideen, R. Zandi, Phys. Rev. B **86**, 115449 (2012).
- [12] L.L. De Raad Jr., K.A. Milton, Ann. Phys. **136**, 229 (1981).
- [13] P. Gosdzinsky, A. Romeo, Phys. Lett. B **441**, 265 (1998).
- [14] G. Lambiase, V.V. Nesterenko, M. Bordag, J. Math. Phys. **40**, 6254 (1999).
- [15] K.A. Milton, A.V. Nesterenko, V.V. Nesterenko, Phys. Rev. D **59**, 105009 (1999).
- [16] I. Cavero-Peláez, K.A. Milton, J. Phys. A **39**, 6225 (2006); A. Romeo, K.A. Milton, J. Phys. A **39**, 6225 (2006); I. Brevik, A. Romeo, Physics Scripta **76**, 48 (2007).
- [17] A.A. Saharian, Izv. AN Arm. SSR. Fizika **23**, 130 (1988) [Sov. J. Contemp. Phys. **23**, 14 (1988)].
- [18] A.A. Saharian, Dokladi AN Arm. SSR **86**, 112 (1988) (Reports NAS RA, in Russian); F.D. Mazzitelli, M.J. Sanchez, N.N. Scoccola, J. von Stecher, Phys. Rev. A **67**, 013807 (2002); K. Tatur, L. M. Woods, I. V. Bondarev, Phys. Rev. A **78**, 012110 (2008).
- [19] A.A. Saharian, "The Generalized Abel-Plana Formula. Applications to Bessel Functions and Casimir Effect," Report No. IC/2000/14 (hep-th/0002239).
- [20] A. Romeo, A.A. Saharian, Phys. Rev. D **63**, 105019 (2001).
- [21] A.A. Saharian, A.S. Tarloyan, J. Phys. A **39**, 13371 (2006).
- [22] K. Tatur, L.M. Woods, Phys. Lett. A **372**, 6705 (2008).
- [23] D.A.R. Dalvit, F.C. Lombardo, F.D. Mazzitelli, R. Onofrio, Europhys. Lett. **67**, 517 (2004); F.D. Mazzitelli, D.A.R. Dalvit and F.C. Lombardo, New. J. Phys. **8**, 240 (2006); D.A.R. Dalvit, F.C. Lombardo, F.D. Mazzitelli, R. Onofrio, Phys. Rev. A **74**, 020101(R) (2006); S.J. Rahi, T. Emig, R.L. Jaffe, M. Kardar, Phys. Rev. A **78**, 012104 (2008); A.W. Rodriguez, J.N. Munday, J.D. Joannopoulos, F. Capasso, D.A.R. Dalvit, S.G. Johnson, Phys. Rev. Lett. **101**, 190404 (2008); S.J. Rahi, T. Emig, N. Graham, R.L. Jaffe, M. Kardar, Phys. Rev. D **80**, 085021 (2009); M. Bordag, V. Nikolaev, J. Phys. A **42**, 415203 (2009); F.C. Lombardo, F.D. Mazzitelli, P.I. Villar, D.A.R. Dalvit, Phys. Rev. A **82**, 042509 (2010); E. Noruzifar, T. Emig, R. Zandi, Phys. Rev. A **84**, 042501 (2011); E. Noruzifar, T. Emig, U. Mohideen, R. Zandi, Phys. Rev. B **86**, 115449 (2012).

- [24] A.R. Kitson, A. Romeo, Phys. Rev. D **74**, 085024 (2006); J.P. Straley, G.A. White, E.B. Kolomeisky, Phys. Rev. A **87**, 022503 (2013); N. Graham, Phys. Rev. D **87**, 105004 (2013); J.P. Straley, G.A. White, E.B. Kolomeisky, arXiv:1403.3439.
- [25] V.N. Marachevsky, Phys. Rev. D **75**, 085019 (2007).
- [26] V.V. Nesterenko, G. Lambiase, G. Scarpetta, J. Math. Phys. **42**, 1974 (2001); A.H. Rezaeian, A.A. Saharian, Class. Quantum Grav. **19**, 3625 (2002); A.A. Saharian, A.S. Tarloyan, J. Phys. A **38**, 8763 (2005); A.A. Saharian, Eur. Phys. J. C **52**, 721 (2007); A.A. Saharian, A.S. Tarloyan, Annals Phys. **323**, 1588 (2008); I. Brevik, S.A. Ellingsen, K. A. Milton, Phys. Rev. E **79**, 041120 (2009); S.A. Ellingsen, I. Brevik, K.A. Milton, Phys. Rev. E **80**, 021125 (2009); K.A. Milton, J. Wagner, K. Kirsten, Phys. Rev. D **80**, 125028 (2009); S.A. Ellingsen, I. Brevik, K.A. Milton, Phys. Rev. D **81**, 065031 (2010).
- [27] M. Bordag, Phys. Rev. D **73**, 125018 (2006); H. Gies, K. Klingmuller, Phys. Rev. D **74**, 045002 (2006); T. Emig, R.L. Jaffe, M. Kardar, A. Scardicchio, Phys. Rev. Lett. **96**, 080403 (2006); G.L. Klimchitskaya, E.V. Blagov, V.M. Mostepanenko, J. Phys. A **39**, 6481 (2006); K.A. Milton, J. Wagner, Phys. Rev. D **77**, 045005 (2008); S.J. Rahi, A.W. Rodriguez, T. Emig, R.L. Jaffe, S.G. Johnson, M. Kardar, Phys. Rev. A **77**, 030101 (R) (2008); M.T.H. Reid, A.W. Rodriguez, J. White, S.G. Johnson, Phys. Rev. Lett. **103**, 040401 (2009); N. Graham, A. Shpunt, T. Emig, S.J. Rahi, R.L. Jaffe, M. Kardar, Phys. Rev. D **81**, 061701 (2010); M. Schaden, Phys. Rev. A **82**, 022113 (2010); R.S. Decca, E. Fischbach, G.L. Klimchitskaya, D.E. Krause, D. Lopez, V.M. Mostepanenko, Phys. Rev. A **82**, 052515 (2010); A. Weber, H. Gies, Phys. Rev. D **82**, 125019 (2010); N. Graham, A. Shpunt, T. Emig, S.J. Rahi, R.L. Jaffe, M. Kardar, Phys. Rev. D **83**, 125007 (2011); L.P. Teo, Phys. Rev. D **84**, 065027 (2011); L.P. Teo, Phys. Rev. D **84**, 025022 (2011); P. Rodriguez-Lopez, T. Emig, Phys. Rev. A **85**, 032510 (2012); L.P. Teo, Phys. Rev. D **87**, 045021 (2013).
- [28] I. Brevik, T. Toverud, Class. Quantum Gravity **12**, 1229 (1995); E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, A.S. Tarloyan, Phys. Rev. D **74**, 025017 (2006); E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, Phys. Lett. B **645**, 245 (2007); E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, A.S. Tarloyan, Phys. Rev. D **78**, 105007 (2008); V.V. Nesterenko, I.G. Pirozhenko, Class. Quantum Grav. **28**, 175020 (2011).
- [29] M.R. Setare, R. Mansouri, Classical Quantum Gravity **18**, 2331 (2001); A.A. Saharian, T.A. Vardanyan, Classical Quantum Gravity **26**, 195004 (2009); E. Elizalde, A.A. Saharian, T.A. Vardanyan, Phys. Rev. D **81**, 124003 (2010); A.A. Saharian, Int. J. Mod. Phys. A **26**, 3833 (2011); P. Burda, JETP Lett. **93**, 632 (2011); S. Bellucci, A.A. Saharian, A.H. Yeranyan, arXiv:1402.6997.
- [30] K.A. Milton, A.A. Saharian, Phys. Rev. D **85**, 064005 (2012).
- [31] N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
- [32] A.A. Saharian, Izv. AN Arm. SSR, Matematika **22**, 166 (1987) [Sov. J. Contemp. Math. Anal. **22**, 70 (1987)].
- [33] A. A. Saharian, *The Generalized Abel-Plana Formula with Applications to Bessel Functions and Casimir Effect* (Yerevan State University Publishing House, Yerevan, 2008); Report No. ICTP/2007/082; arXiv:0708.1187.
- [34] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.

- [35] F.W.J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974);
T.M. Dunster, SIAM J. Math. Anal. **21**, 995 (1990).
- [36] A.A. Saharian, Phys. Rev. D **70**, 064026 (2004).
- [37] *Handbook of Mathematical Functions*, edited by M. Abramowitz, I.A. Stegun (Dover, New York, 1972).